- **5352:** *Proposed by Arkady Alt, San Jose, CA*

Evaluate $\sum\limits_{k=0}^{n} x^k-(x-1)\sum\limits_{k=0}^{n-1}(k+1)x^{n-1-k}$.

Solution 1 by G.C. Greubel, Newport News, VA

Consider the series $\displaystyle\sum_{k=0}^{n} x^k=\frac{1-x^{n+1}}{1-x}$ for which the series in question becomes

$$\begin{aligned}
S &= \sum_{k=0}^{n} x^k-(x-1)\sum_{k=0}^{n-1}(k+1)\,x^{n-k-1} \\
&= \frac{1-x^{n+1}}{1-x}+(1-x)\left[\sum_{k=0}^{n-1} x^{n-k-1}+\sum_{k=0}^{n} k\,x^{n-k-1}\right] \\
&= \frac{1-x^{n+1}}{1-x}+(1-x)\,x^{n-1}\left[\frac{1-\left(\frac{1}{x}\right)^n}{1-\frac{1}{x}}+x\,\partial_x\left(\frac{1-\left(\frac{1}{x}\right)^n}{1-\frac{1}{x}}\right)\right] \\
&= \frac{1-x^{n+1}}{1-x}+(1-x)\cdot\frac{1-x^n}{1-x}+(x-1)\,x^{n+2}\left[\frac{n(x-1)+1-x^n}{x^{n+2}}\right] \\
&= \frac{1-x^{n+1}}{1-x}+1-x^n+n-\frac{1-x^n}{1-x} \\
&= n+1.
\end{aligned}$$

From this it can be stated that

$$\sum_{k=0}^{n} x^k-(x-1)\sum_{k=0}^{n-1}(k+1)\,x^{n-k-1}=n+1.$$

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Since $\displaystyle\sum_{k=0}^{n-1}(k+1)x^{n-1-k}=\sum_{k=1}^{n} kx^{n-k}$, then $\displaystyle(x-1)\sum_{k=0}^{n-1}(k+1)x^{n-1-k}=$
$\displaystyle\sum_{k=1}^{n} kx^{n-k+1}-\sum_{k=2}^{n+1}(k-1)x^{n-k+1}=x^n+\sum_{k=2}^{n}x^{n-k+1}-n=-n+\sum_{k=1}^{n}x^k$, and therefore

$$\sum_{k=0}^{n} x^k-(x-1)\sum_{k=0}^{n-1}(k+1)x^{n-1-k}=1+n.$$

Solution 3 by Henry Ricardo, New York Math Circle, NY

Denote the given expression as $F_n(x)$, where we assume that $n\geq 1$ and $x\neq 0$. Since $F_1(x)=1+x-(x-1)(0)=2=1+1$ and $F_2(x)=(1+x+x^2)-(x-1)(x+2)=3=2+1$, we conjecture that $F_n(x)=n+1$ for all nonzero values of x and prove this by induction.

Suppose that $F_N(x) = N+1$ for some integer $N \geq 3$ and all $x \neq 0$. Then

$$\begin{aligned}
F_{N+1}(x) &= \sum_{k=0}^{N+1} x^k - (x-1)\sum_{k=0}^{N}(k+1)x^{N-k} \\
&= x\sum_{k=0}^{N} x^k + 1 - (x-1)\left(\sum_{k=0}^{N-1}(k+1)x^{N-k} + N + 1\right) \\
&= x\sum_{k=0}^{N} x^k + 1 - (x-1)\left(x\sum_{k=0}^{N-1}(k+1)x^{N-k-1} + N + 1\right) \\
&= 1 + x\left(\sum_{k=0}^{N} x^k - (x-1)\sum_{k=0}^{N-1}(k+1)x^{N-k-1}\right) - (N+1)(x-1) \\
&= 1 + x(N+1) - (N+1)(x-1) = N+2 = (N+1)+1.
\end{aligned}$$

Also solved by Dionne T. Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Jerry Chu (student, Saint George's School), Spokane, WA; Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student, Taylor University), Upland, IN; Ed Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, SUNY College at Oneonta, Oneonta, NY; Haroun Meghaichi (student, University of Science and Technology, Houari Boumediene), Algiers, Algeria; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Henry Ricardo (two additional solutions to his one above), New York Math Circle, New York; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins of Georgia Southern University in Statesboro, GA, and the proposer.

5353: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n with complex coefficients. Prove that all its zeros lie in the disk $\mathcal{D} = \{z \in C : |z| < r\}$, where

$$r = \left\{1 + \left(\sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|^3\right)^{1/2}\right\}^{2/3}$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

$A(z)$ is a polynomial of degree n. So $a_n \neq 0$. Let $|z| \geq r$. Then, by Hölder's inequality,

$$\frac{1}{|a_n|}|A(z)| \geq |z|^n - \sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right||z|^k \geq |z|^n - \left(\sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|^3\right)^{\frac{1}{3}}\left(\sum_{k=0}^{n-1}|z|^{\frac{3k}{2}}\right)^{\frac{2}{3}} = |z|^n - \left(r^{\frac{3}{2}}-1\right)^{\frac{2}{3}}\left(\frac{|z|^{\frac{3n}{2}}-1}{|z|^{\frac{3}{2}}-1}\right)^{\frac{2}{3}}$$

$$\geq |z|^n - \left(r^{\frac{3}{2}}-1\right)^{\frac{2}{3}}\left(\frac{|z|^{\frac{3n}{2}}-1}{|r|^{\frac{3}{2}}-1}\right)^{\frac{2}{3}}$$

$$= |z|^n - \left(|z|^{\frac{3n}{2}} - 1\right)^{\frac{2}{3}} > |z|^n - \left(|z|^{\frac{3n}{2}}\right)^{\frac{2}{3}} = 0.$$

So all zeros lie in the open disk $\mathcal{D}$

Solution 2 by Kee-Wai Lau, Hong Kong, China

According to Theorem (27.4) on p. 124 of [1], we have the following result:

For any p and q such that $p > 1, q > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1$, the polynomial $f(x) = a_0 + a_1 x + \cdots a_n z^n, a_n \neq 0$ has all of its zeros in the circle

$$|z| < \left\{1 + \left(\sum_{j=0}^{n-1} \left|\frac{a_j}{a_n}\right|^p\right)^{q/p}\right\}^{1/q} \leq \left(1 + n^{q/p} M^q\right)^{1/q},$$

where $M = \max \left|\dfrac{a_j}{a_n}\right|, j = 0, 1, \cdots, n-1$.

In particular, when $p = 3$, the result of the above problem follows.

Reference: 1. M. Marden: *Geometry of Polynomials,* Mathematical Surveys and Monographs Number 3, American Mathematical Society, (1966).

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

- **5354:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a, b, c > 0$ be real numbers. Prove that the series

$$\sum_{n=1}^{\infty} \left[n \cdot \left(a^{\frac{1}{n}} - \frac{b^{\frac{1}{n}} + c^{\frac{1}{n}}}{2}\right) - \ln \frac{a}{\sqrt{bc}}\right],$$

converges if and only if $2\ln^2 a = \ln^2 b + \ln^2 c$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Let x be real. By Taylor's theorem there is a number $h = h(x),\ 0 \leq h \leq 1$, such that $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} e^{hx}$. We choose $x = \dfrac{\ln a}{n},\ x = \dfrac{\ln b}{n},\ x = \dfrac{\ln c}{n}$ and get

$$a^{\frac{1}{n}} = 1 + \frac{\ln a}{n} + \frac{\ln^2 a}{2n^2} + \frac{\ln^3 a}{6n^3} a^{\frac{h}{n}},\ 0 \leq h = h(a, n) \leq 1,$$

$$b^{\frac{1}{n}} = 1 + \frac{\ln b}{n} + \frac{\ln^2 b}{2n^2} + \frac{\ln^3 b}{6n^3} b^{\frac{h}{n}},\ 0 \leq h = h(b, n) \leq 1,$$

$$c^{\frac{1}{n}} = 1 + \frac{\ln c}{n} + \frac{\ln^2 c}{2n^2} + \frac{\ln^3 c}{6n^3} c^{\frac{h}{n}},\ 0 \leq h = h(c, n) \leq 1.$$